A Brief Synopsys

of

The Millennium Problems: The Seven Greatest Unsolved Mathematical Puzzles of Our Time

by

Keith Devlin

Aaron Seefeldt

Math 629 – History of Mathematics

Dr. S. A. Fulling

October 9, 2006

Part 1: Introduction

Just over six years ago a monolithic and audacious challenge was issued by the Clay Mathematics Institute (CMI). In its May 2000 conference, the CMI announced that $1 million would be offered for a solution to any one of the seven problems that a small panel of acclaimed mathematicians had selected. These problems span many areas of mathematics and are regarded as being among most challenging and important problems that remain unproven. The timing of this challenge was deliberate, since exactly 100 years before, the German mathematician David Hilbert had made a similar boast when he published a list of 23 significant mathematical problems that needed to be solved for the 20th century.

Devlin's book, The Millennium Problems: The Seven Greatest Unsolved Mathematical Puzzles of Our Time, essentially develops the history of each problem and the mathematics involved, why they are difficult, and why mathematicians regard it as important. However, one cannot simply just dive into the problems themselves, since they require a profound understanding due to their extreme level of abstraction. The problems themselves delve into deep abstractions that have been progressively developed (in some cases spanning centuries), and thus even the "experts" struggle to understand them.

Throughout the course of this paper, I have tried to summarize the history behind each problem and illustrate some of the mathematics involved. The problems are listed in what could be considered "easiest to hardest" to understand, although that notion is highly subjective. My motivation for selecting this book for review was threefold. First, it exposes the reader to ongoing problems that remain viable. Second, all of the problems have an element of historical development (some more than others). Third, the scope of the mathematics involved is both broad and deep, and as such this book quickly separates itself from other such popularized books.

Part 2: The 7 Millennium Problems

A. The Riemann Hypothesis

Regarded as "the most important unsolved problem of mathematics" (p. 19), the Riemann hypothesis was originally proposed in an 1859 paper on analytic number theory that Riemann had published. Interestingly, this is the only problem on the CMI list that was also on Hilbert's 1900 list. But who was Riemann, and what is the nature of his legacy?

Bernhard Riemann was born into a poor German family in the small town of Breslenz in 1826. The son of a Lutheran pastor, Riemann suffered from poor health his entire life and most biographers agree that he was an acute hypochondriac and obsessed with perfection. However, his mathematical prowess was quickly recognized. For example, while attending high school in Luneburg, one of his teachers gave him a copy of Legendre's *Number Theory*, a 900 page text. Less than a week later, Riemann gave it back remarking, "This is a wonderful book; I know it by heart." (p. 24)

Although Riemann's father insisted that he pursue a theological career, Riemann fortunately abandoned this notion upon entering the university at Gottingen.

Interestingly, Gauss himself was a faculty member there at the time, although Gauss' career was in its waning years. Riemann transferred to the University of Berlin and studied under the likes of Jacbi, Dirichlet, and Klein. He returned to Gottingen in 1849 and his doctoral work was supervised by Gauss. In 1851 he received his doctorate in mathematics and was quickly assimilated into academia. By 1859 Riemann attained the rank of full professor at Gottingen. He died in 1866 in Italy – a victim to the ubiquitous scourge of tuberculosis that plagued the 19th century.

It is important to note that around the time that Riemann was doing his doctoral work, there was a mathematical revolution going on that Dirichlet had initiated. This revolution was actually a transformation from the "old computational approach/algorithmic view" to a "conceptual, abstract approach" (p. 26). Cauchy's development of the epsilon-delta definitions of continuity and differentiability are a hallmark of this revolution in abstract conceptualism. Riemann later characterized Cauchy's work as a "turning point in the conception of the infinite" (p. 27). This conceptualism, combined with Riemann's deep mathematical intuition, was instrumental to his success, as we will see below.

The Riemann Hypothesis is all about prime numbers and owes much to the contributions of his predecessors. The story begins around the 4th century BC, when Euclid first proved that there are infinitely many primes. Although Euclid's proof is not listed here, we will shortly return to the notion of the infinitude of the prime numbers. Next, fast-forward to the 18th century. While he was studying the distribution of prime numbers, Gauss discovered something about the density of the primes. Let $N \in \mathbf{Z}^+$, and define the prime density function to be

$$D_N = \frac{P(N)}{N} \tag{1}$$

where $P(N)$ is the number of primes less than or equal to N. What Gauss discovered was that for large N, we have

$$D_N = \frac{P(N)}{N} \approx \frac{1}{\ln(N)} \tag{2}$$

This means that there is a "systematic pattern to the way that [primes] thin out" (p. 31).

Returning to the infinitude of primes, in 1740 Euler offered another proof that was far different from Euclid's, but pivotal to the Riemann Hypothesis. Here's how Euler did it: Assume that the number of primes is finite, and let them be arranged $p_1 = 2$, $p_2 = 3$, …, p_n. Since $0 < \frac{1}{p_i} < 1$ for all i, the geometric series

$\sum_{k=0}^{\infty} \frac{1}{(p_i)^k}$ must converge to $\frac{1}{1-\frac{1}{p_i}}$. Thus,

$$\prod_{i=1}^{n}\left(\sum_{k=0}^{\infty}\frac{1}{(p_i)^k}\right) = \prod_{i=1}^{n}\frac{1}{1-\frac{1}{p_i}} \tag{3}$$

We can expand the left hand side by the Fundamental Theorem of Arithmetic, which gives

$$\prod_{i=1}^{n}\left(\sum_{k=0}^{\infty}\frac{1}{(p_i)^k}\right)=\left(1+\frac{1}{2}+\frac{1}{4}+...\right)\left(1+\frac{1}{3}+\frac{1}{9}+...\right)\left(1+\frac{1}{5}+\frac{1}{25}+...\right)\cdots$$

$$\times\left(1+\frac{1}{p_i}+\frac{1}{p_i^2}+...\right)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+...=\sum_{m=1}^{\infty}\frac{1}{m}$$

However, this is the harmonic series, which diverges. Since $\prod_{i=1}^{n}\frac{1}{1-\frac{1}{p_i}}$ is finite, we have a contradiction. Therefore, there must be infinitely many primes.

Now, because Euler was interested in proving whether or not the "prime harmonic series" diverges (which it does), he defined the following function:

$$\zeta(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+...=\sum_{n=1}^{\infty}\frac{1}{n^s} \tag{4}$$

This is known as the Euler zeta function. It is finite if and only if $s>1$. Euler then proceeded to split the sum into two parts: the prime terms and the composite terms:

$$\zeta(s)=\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{5^s}+...\right)+\left(\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+...\right) \tag{5}$$

Euler also discovered another way to define his zeta function:

$$\zeta(s)=\frac{1}{1-\left(\frac{1}{2}\right)^s}\times\frac{1}{1-\left(\frac{1}{3}\right)^s}\times\frac{1}{1-\left(\frac{1}{5}\right)^s}\times...=\prod_{p\ prime}^{\infty}\frac{1}{1-\left(\frac{1}{p}\right)^s} \tag{6}$$

Then, in 1859 Riemann basically picked up where Euler left off. Mostly due to Gauss, complex numbers had achieved general acceptance in the esoteric mathematical community. Using his conceptual intuition, Riemann replaced the real number s in Euler's zeta function with a complex number z. Riemann then used his "new" zeta function $\zeta(z)$ to explore the pattern of primes in relation to the density function above. Specifically, he was interested in solutions to

$$\zeta(z)=0 \tag{7}$$

What Riemann discovered was that the numbers -2, -4, -6, … were all "zeros" to his zeta function. He later generalized his result that all other zeros had the form $z=\frac{1}{2}+bi$ where b is a real number. The Riemann Hypothesis says that "there are infinitely many other complex numbers z for which $\zeta(z)=0$ all lie on the critical line $\mathrm{Re}(z)=\frac{1}{2}$" (p. 46).

The reason why the Riemann Hypothesis is so important is that it tells us a great deal about the fundamental nature of prime numbers and how they are distributed. It is widely believed that if it is proved, then that proof will lead to advanced factoring methods for large composite numbers. Because much of modern cryptography (e.g.

the RSA scheme) is based on the inability of computers to factor large composite numbers quickly, a proof of the Riemann Hypothesis could threaten the security offered by encryption based on prime numbers. Although it has been proved true for the first 1.5 billion zeros, all that can be said at this point is that Riemann's hypothesis is *probably* true.

B. The Mass Gap Hypothesis

The second Millennium Problem is historically one of the oldest, since its origins go back as far as ancient Greece. It is also a problem that highlights the never ending struggle between physicists, who seek accurate theoretical descriptions of reality that agree with observations, and the mathematicians, who tend to neglect natural phenomena and focus on a more pure form of mathematics. The second problem can be stated as follows: "Prove that for any compact, simple gauge group, the quantum Yang-Mils equations in four-dimensional Euclidean space have a solution that predicts a mass gap" (p. 94).

In order to answer the challenge issued by the physicists, the mathematicians will have to develop new areas of mathematics so that they can "catch up" to the physics. This quandary is similar to how quantum theory required the development of functional analysis and group representation theory. But this is not the first time that math has lagged behind physics, since it took nearly two centuries after Newton and Liebnitz to fully explain *why* calculus worked.

The connection between electricity and magnetism was first made in 1820 by Oersted, and in 1821 by Ampere, who were both doing crude experiments with electric currents, magnetic needles, and wires. Later, Lord Kelvin suggested that "electricity and magnetism arose from some form of force field in the ether" (p. 73). At that time, it was widely accepted that the either was an invisible jelly-like medium that permeated all of space. (The ether concept was later discarded after the failure, or success rather, of the Michelson-Morley experiment in 1887.)

Then in 1865, Maxwell published his work on electrodynamics, and successfully calculated the speed of electromagnetic radiation to be about $3\times10^8\ m/s$. It should be noted that Maxwell's equations are really just Gauss's law, Faraday's law, Ampere's law with Maxwell's correctional term, and $\nabla\cdot\mathbf{B}=0$, or no magnetic monopoles. Later, in 1905, Einstein published his theory of special relativity, which is essentially two postulates: the laws of physics are valid in all inertial (non-accelerating) reference frames, and that the speed of light is the same for all observers in inertial reference frames. Einstein followed up in 1915 with general relativity, which described the nature of mass, light, and gravity, and was confirmed in a 1919 when an astronomer observed the bending of light during a solar eclipse. However, general relativity is not a quantum theory.

In the early part of the 20th century, the development of quantum theory dissolved the "solar-system" explanation of the atom, and gave rise to the particle-wave duality notion of light and sub-atomic particles. Also, in 1900 Planck's description of the discrete nature of energy led to the discovery that light is made up of discrete packets of energy (photons) whose energy is proportional to the light wave's frequency.

According to the Standard Model, it is believed that there are 4 fundamental forces of nature (the strong and weak nuclear forces, the electromagnetic force, and gravity). Thus, any complete theory of matter must successfully incorporate and connect all four. What is needed is a *quantum field theory*, where the classical fields are quantized and the notion of particles as localized points is abandoned in favor of a field-like characterization. To arrive at such a theory, or *grand unified theory* (GUT), they key turns out to be the concept of symmetry.

"The set of all the symmetries of a given object is called the symmetry group for that object" (p. 89). The "object" in question can in actuality be abstract, such as the equation of a field. A key step was made when physicists noticed that the conservation laws (like energy and momentum) are really the result of some symmetry, and each conservation law has an associated group.

Now, in general, a collection of "entities," G, "and an operation $*$ that combines any two elements x and y in the set G to give a further element $x*y$ in G, then that collection is called a group if the following conditions are true:

G1. For all $x, y, z \in G$, $(x*y)*z = x*(y*z)$

G2. There exists an $e \in G$ such that $x*e = e*x = x$ for all $x \in G$

G3. For each element $x \in G$, there is an element $y \in G$ such that $x*y = y*x = e$

G4. For all $x, y \in G,\ x*y = y*x$

Note that groups that satisfy G4 (the commutative axiom) are referred to as abelian, and if they do not commute, nonabelian. In this sense, the G4 condition does not have to be satisfied in order for the entities to be classified as a group.

Now, Maxwell's equations are "invariant with respect to a change of scale" (p. 90). In 1919, the physicist Weyl discovered that Maxwell's equations "keep their form even if the electromagnetic potentials are multiplied by quantum-mechanical phase factors, or gauges" (p. 90). The focus on phase then classified Maxwell's equations into the one-dimensional unitary group U(1). Weyl thus gave birth to the notion of using symmetry to connect the four fundamental forces.

The quantum description of electromagnetism was developed in the 1940s and became known as quantum electrodynamics (QED). Later, in 1954 Yang and Mills replaced the U(1) group with a compact Lie group in their nonabelian gauge theory. The general impetus for this was trying to find the right gauge group that would connect the two nuclear forces (the weak and strong force) with the more elusive gravity. But there were problems with classical Yang-Mills theory, such as the gauge and scale anomaly.

Recently, the theory known as quantum chromodynamics (QCD) was developed. It essentially describes how the strong interaction is carried by gluons (massless bosons). One of its peculiar properties is called asymptotic freedom, which means that quarks and gluons interact very weakly. QCD also has two other key features. First, there must be a "mass gap," i.e. a nonzero minimum energy level for excitations of the vacuum. Second, this notion of confinement, i.e. that the force between quarks does not diminish as they are separated, which explains why there are no free quarks.

Proving the Mass Gap Hypothesis would essentially allow us to finally make sense of the standard model of particle physics, which at times seems like a mosaic of complicated and intertwined theories.

C. The P versus NP Problem

Although the third problem is considered (by some) to be the easiest of the seven Millennium Problems, it is nevertheless quite a complicated matter. For the origins of this one, we return to David Hilbert, who around the turn of the century began the bold, and erroneous quest known as the Hilbert Program. This program was basically an attempt to write down a set of axioms (fundamental assumptions that are "true") within a branch of mathematics, from which "all of the facts in that branch" (p. 108) could be deduced. In 1931, Godel proved this was impossible, since "there will always be some true statements that cannot be proved from those axioms" (p. 108). Godel developed the concept of a "computable function," which arose form considerations about what computations could in theory be carried out.

In the early 1960s, a young graduate student at Harvard named Stephen Cook was working on his Ph.D. in mathematics. Although his real passion was the fledgling discipline of computer science, at the time it was still buried in mathematics and not well established. He completed his dissertation in 1966, and in 1971 published a paper entitled *The Complexity of Theorem Proving Procedures* that gave birth to the third Millennium Problem. In that paper, Cook introduced the concept of NP completeness.

The Traveling Salesman Problem, first introduced by Karl Menger in the 1930s, provides a good transition into the idea of NP completeness. Suppose a salesman had to visit 10 towns, and wanted to know the shortest distance he would have to travel. Although the math itself is trivial, the number of calculations (10!) is not. Although you can try for approximate answers, there really is now way to effectively solve this problem if the number of cities is large, even using an advanced computer.

Key to this entire idea is the number of computations involved in a given problem, since that number will determine how long a computer takes to solve it. It turns out that all "of the four arithmetic operations – addition, subtraction, multiplication, and division – are polynomial time processes" (p. 120). When faced with some computational process, mathematicians look for "an algebraic expression" (such as $cN, cN^2, or\ cN^k$) known as the time complexity function that gives an upper estimate for the number of calculations that will be involved in order to know how long it will take.

In general, polynomial time processes (called type P) like the expressions above are the only ones that a computer can effectively handle. In contrast, exponential time processes are generally more than even a sophisticated computer can handle. For example, consider the two time complexity functions N^3 and 3^N, and assume you are using a computer that can perform 1 million arithmetic calculations per second. If N=50, then for N^3 it would only take the computer .125 seconds to complete. For 3^N, it would take the computer 2×10^8 *centuries* to compute!

Another class of time processes was introduced in the 1960s called NP. A problem is type NP "if it can be solved or completed in polynomial time…by a computer that is able to make a random choice between a range of alternatives and does so with perfect luck" (p. 124). Because so many real world problems have exponential time processes (like the Traveling Salesman), the NP class allowed mathematicians to deal with them more tractably, or so it seemed.

In his 1971 paper, Cook was working with one particular NP problem. What he did was to show that if his NP problem could be solved with polynomial time processes, then so could any NP problem (he showed how to equivocate any NP problem into the particular one he was working on). To say that a problem is NP complete means that it cannot be solved by polynomial time processes. If you had a problem that was NP complete and you were able to solve it with polynomial time processes, then what Cook did was to show that every NP problem could then be solved with polynomial time processes.

However, this is merely a conjecture. It may turn out that P=NP after all, but as of yet nobody has shown offered any NP problem that can be solved in polynomial time. Realizing that a problem you are working on is NP complete is not a total disaster, however, because that realization would then immediately force you to use approximation methods, or abandon the problem all together and save valuable time.

D. The Navier-Stokes Equations

The story of the fourth Millennium Problem goes back to 1822, when Claude Louis Henri Navier was teaching mathematics at College of Bridges and Causeways in France. Perhaps due to his training as an engineer, Navier decided to improve and build upon Euler's equations that described the flow of a viscosity-free fluid. Navier, however, allowed the liquid to have some viscosity (more realistic), and though his reasoning was flawed, by some measure of luck he arrived at the current version of the Navier-Stokes equations.

Twenty years later, an Irish mathematician named George Stokes was carrying out research at Cambridge. Using the methods of calculus from the start, Stokes was able to independently and correctly derive Navier's equations. Although it appeared that the story of fluid flow was nearly complete by the mid 19th century, nobody has been able to solve these Navier-Stokes equations, or even show if a solution exists.

But the real story began earlier, due mostly in part to the development of calculus and the study of continuous motions. Just like its linear analog, Daniel Bernoulli regarded a "continuous fluid as made up of infinitesimally small discrete regions, infinitesimally close together" (p. 150). His efforts culminated in the 1738 publishing of his book Hydrodynamics. Euler then built on Bernoulli's work and formulated his equations for frictionless flow, but was unable to solve them. Here is how Euler proceeded:

Assume that each point $P=(x,y,z)$ in the fluid is subject to the time dependent forces $f_x(x,y,z,t), f_y(x,y,z,t), f_z(x,y,z,t)$, or $\mathbf{f}=(f_x,f_y,f_z)$. Let the velocity of the fluid at P be $u_x(x,y,z,t)$ and likewise for u_y and u_z, or more compactly $\mathbf{u}=(u_x,u_y,u_z)$. (Euler assumed that these functions are well behaved). The fluid is also assumed to be incompressible, which is equivalent to saying the divergence of the velocity is zero, or $\nabla\cdot\mathbf{u}=0$. Let $\mathbf{p}=p(x,y,z,t)$ be the pressure in the fluid at the point P at time t. Using Newton's law $\mathbf{F}=\frac{d\mathbf{p}}{dt}$ (this **p** is momentum), Euler was able to arrive at

$$\frac{\partial\mathbf{u}}{\partial t}+(\nabla\cdot\mathbf{u})\mathbf{u}=\mathbf{f}-\nabla\mathbf{p} \qquad (8)$$

Now, in order to allow for viscosity, Navier and Stokes introduced the constant $v>0$ and the viscous force to the right hand side of equation (8) to obtain:

$$\frac{\partial\mathbf{u}}{\partial t}+(\nabla\cdot\mathbf{u})\mathbf{u}=\mathbf{f}-\nabla\mathbf{p}+v\nabla^2\mathbf{u} \qquad (9)$$

This is the most general form of the Navier-Stokes equations, and they have never been solved, nor has Euler's "easier" zero-viscosity form. Although (9) can be solved in two dimensions, this provides no clue as to what happens in three dimensions. It seems ironic that the flow of fluids is all around us, from brushing your teeth to watering the lawn, but the complete solution to describe that flow has remained elusive for centuries.

E. The Poincare Conjecture

For the fifth Millennium Problem we turn to the branch of math known as algebraic topology and the life of Henri Poincare. Born in 1854 in France into an affluent and politically active family, Poincare became a kind of Carl Sagan in 19th century France in terms of his publications on popular science. Biographers agree that Poincare often linked ideas in visual ways and approached math through a conceptual based approach.

After graduating from the Ecole Polytechnique in 1875, Poincare finished his doctorate under Hermite at the University of Paris. He remained in academia until his death in 1912 at the age of 58. One of the interesting quirks about Poincare was that he believed that the brain worked on mathematical problems subconsciously during sleep, so he maintained a very strict sleep schedule. He also rejected Hilbert's axiomatic approach to mathematics. One of his major feats as a mathematician is that he is "generally credited as the originator of the theory of analytic functions of several complex variables" (p. 160).

But it was Poincare's 1895 publication of The Analysis of Position that launched topology into the 20th century. Although Gauss had done dome preliminary work in topology, the impetus was really the struggle to understand why differential calculus worked. The development of a geometry where shapes and figures retain their properties after continuous deformations was key, as well the idea that two points are infinitely close before and after a topological transformation.

Not surprisingly, the first real topological theorem can be attributed to Euler. Euler was studying the now famous problem of the Konigsberg bridges, and trying to determine whether or not there "was a route that traversed each bridge exactly once" (p. 169). Euler discovered that the exact layout of the bridges and islands was not important, but rather the network formed by the bridges was significant. Euler went on to prove that for any network drawn on a flat surface, the following is true:

$$V - E + F = 1 \tag{10}$$

where V is the total number of vertices, E is the number of edges, and F is the total number of faces. This is an example of topological invariance and is referred to as the Euler characteristic. In general, the continuous deformation of a network does not change that network. Another example of invariance is the famous Four Color Theorem in 2 dimesions.

Another invariant of topology is something called orientability, which is property where you cannot turn clockwise into counterclockwise along a surface by sliding shapes around it (and vice versa for non-orientabiltiy). These different invariances have led topologists to classify surfaces based on the Euler characteristic and orientability.

Early in the 20th century, mathematicians began classifying higher-dimensional surfaces called manifolds (note that a sphere is 2-dimensional since its surface is 2-dimensional, even though the object itself is 3-D). Thus the 3-manifolds (or hypersurfaces), as they are called, are the 3-dimensional analog of surfaces. A surface "on which every loop can be shrunk down to a point without leaving the surface is topologically equivalent to a sphere" (p. 185).

What Poincare asked that was so startling was this: "Is it possible that a 3-manifold can have the loop-shrinking property and not be equivalent to a 3-sphere?" (p. 186). This is the so called Poincare conjecture. It has been proven true for every dimension except 3, but Poincare's original conjecture remains unproven.

F. The Birch and Swinnerton-Dyer Conjecture

The sixth Millennium Problem arose during the early 1960s when computers were still in their infancy. Brian Birch and Peter Swinnerton-Dyer were then faculty members at Cambridge using one of most advanced computers at the time called EDSAC. The asserted their famous conjecture while studying patterns obtained from data involving elliptic curves.

In general, an elliptic curve is of the form

$$y^2 = x^3 + ax + b \quad \text{where } a, b \in \mathbf{Z} \tag{11}$$

If $x^3 + ax + b < 0$ then y will have imaginary roots and the graph will consist of two separate parts. What Birch and Swinnerton-Dyer did was to try to "count" the number of rational points on elliptic curves using modular arithmetic. A tricky point, however, is how do you count an infinite set?

Essentially what they did was to use prime numbers to count the number of integer pair solutions (x, y) modulo p such that

$$y^2 \equiv x^3 + ax + b \pmod{p} \tag{12}$$

Since they used a prime modulus, the counting is finite. Both of them assumed that "the existence of lots of solutions to the congruences for lots of primes would imply that the original equation has infinitely many rational solutions" (p. 198). So how do you find out if there are many solutions to these many congruences?

To begin, let N_p be the number of solutions modulo p to equation (12). They then calculated density functions:

$$\prod_{p \le M,\ p\ prime} \frac{p}{N_p} \tag{13}$$

They examined the data the got for increasing values of M, and tried to "guess" a formula that would describe the data. Birch and Swinnerton-Dyer hypothesized that the original elliptic curve would have an infinite number of rational points if and only if

$$\prod_{p\ prime}^{\infty} \frac{p}{N_p} = 0 \tag{14}$$

This is basically the crux of their illustrious conjecture. Interestingly, there is an analogy to this question that involves right triangles. It turns out that for any rational-sided right triangle with area $d \in \mathbf{Z}$, then the elliptic curve $y^2 = x^3 - d^2 x$ has infinitely many rational points.

Although other mathematicians have restated the Birch and Swinnerton-Dyer conjecture using Taylor polynomials, nobody has been able to prove it yet. Part of the reason this particular problem is so important is because elliptic curves show up in many areas of mathematics and a formal proof would lead to further understanding in areas such as number theory, geometry, and cryptography. To grasp the scope of this problem, it is currently believed that an accurate proof would be several hundred pages and take years to verify by the mathematical community.

G. The Hodge Conjecture

The seventh and final Millennium Problem is regarded as the most difficult not only to understand, but also because it is difficult to state succinctly. The reason is that the Hodge Conjecture is the apex of mathematical abstraction.

Not much is known about William Hodge, but we do know that he was born in Scotland in 1903 and by the age of 33 served on the faculty at Cambridge. One of his biggest contributions was his theories on harmonic integrals. In 1950, Hodge announced his conjecture during a conference at Cambridge and encouraged people to try to solve it. Later writings by Hodge revealed that he thought his conjecture was true.

The Hodge conjecture deals with the calculus of smooth multidimensional surfaces, referred to as a nonsingular projective algebraic variety. These surfaces, or H-objects, and completely abstract, but can be "built up from geometric objects in a purely algebraic way" (p. 219). Now, consider integrals over a generalized path on one of these surfaces. Since "deforming the path leaves the values of such integrals unchanged" (p. 220), these integrals are basically defined on classes of paths.

The Hodge Conjecture says that some of these integrals are zero, "then there's a path in that class that can be described by polynomial equations" (p. 220). Essentially, this conjecture provides a fundamental link between algebraic geometry, topology, and analysis.

But to further understand the conjecture, consider the complex valued function $f(z)$, where z is a complex variable. $f(z)$ can be written as

$$f(z) = u(z) + iv(z) \tag{15}$$

where $u(z)$ and $v(z)$ are real valued functions of the complex variable z. Laplace's equations say that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{16}$$

These equations have been known for some time. What Hodge did was to use the methods of analysis on smooth, complex surfaces, similar to Riemann surfaces. He then realized that certain equations that arose from such surfaces could be viewed as solutions to Laplace's equations.

At present, nobody knows if the conjecture is actually true, and there is no real supportive evidence out there. However, considering Hodges' profound intuition into this area of mathematics (much of which he developed himself), it may very well turn out to be true.

Part 3: Conclusion

It would not be appropriate to summarize the above, since that would effectively amount to a summary of a summary. Rather, I offer some closing comments about Devlin's book. *In toto*, Devlin did a good job of providing broad and readily understandable explanations of the Millennium Problems. For the most part, his book flowed in a logical progression, both within each chapter and as a whole.

However, Devlin's condescending and at times haughty tone pervades the entire book. At times, especially in his discussion of the last three problems, that tone precludes him from providing a more thorough and complete description of the problems. This is probably because he does not fully grasp the concepts himself. He asserts multiple times throughout the book that unless you have Ph.D., then you should not even *attempt* to solve these problems.

Nevertheless, I would still highly recommend this book to anyone who has a sufficient mathematical background. The scope of the book is broad, but that is one of the book's strengths because it exposes the reader (beyond superficially) to many important mathematical concepts. Even if the reader cannot fully grasp the essence of a specific Millennium Problem, Devlin still covers enough other material to be beneficial. However, it would have been helpful if Devlin had included a robust and comprehensive reference list for further reading for motivated readers.

Overall, the book did a good job of blending the history of each problem and the specific mathematician into the story of each chapter. Devlin definitely has somewhat of a flare for mathematical explanation. Some have called his book a "popularization" of deep mathematical issues and have derided it, calling it shallow. This characterization is unfair, because although many of the mathematical details are "trimmed down," the overall intent was to expose the reader to the Millennium Problems and facilitate understanding of abstract concepts.

Part 4: Bibliography

1) "The Millennium Problems: The Seven Greatest Unsolved Mathematical Puzzles of Our Time." Keith Devlin. 2002, Basic Books. ISBN 0-465-01729-0.

2) "Mathematical Journeys." Peter Schumer. 2004, John Wiley & Sons. ISBN 0-471-22066-3.

3) http://claymath.org/millennium/

4) "Introduction to Quantum Mechanics." David Griffiths. 1995, Prentice Hall Inc. ISBN 0-13-124405-1.

5) "Introduction to Electrodynamics." David Griffiths. 1989, 2nd ed. Prentice Hall Inc. ISBN 0-13-481367-7